https://www.linkedin.com/feed/update/urn:li:activity:6482947267509788672

4271. **Proposed by Hung Nguyen Viet**, **supplemented by the Editorial Board**.

a) Let a,b,c be nonzero real numbers such that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1$$

Prove that

$$\sqrt{\frac{(b+c)^2}{a^4}+\frac{(c+a)^2}{b^4}+\frac{(a+b)^2}{c^4}}$$

is a rational function of a,b,c.

b) (Suggested by the Editorial Board). Prove or disprove that the equation

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1$$

has no rational solution.

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.

a) Let $F(a,b,c):=\sqrt{\frac{(b+c)^2}{a^4}+\frac{(c+a)^2}{b^4}+\frac{(a+b)^2}{c^4}}$. Then $F(ka,kb,kc)=\frac{1}{k}F(a,b,c)$ for any positive k.

Since $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1\iff(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)=4$ then $a+b+c\neq 0$ and, therefore, due to homogeneity of equation $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1$ we can prove that $F(a,b,c)$ rational function of a,b,c assuming $a+b+c=1$.

Let $p:=ab+bc+ca,q:=abc$. Then $\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}=4\iff$ $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=4\iff\frac{1+p}{p-q}=4\iff 1+4q=3p\iff q=\frac{3p-1}{4}$ and $p\neq 1/3$ (because $q\neq 0$).

We have $F^2(a,b,c)=\sum\frac{(1-a)^2}{a^4}=\sum\frac{1}{a^4}-2\sum\frac{1}{a^3}+\sum\frac{1}{a^2}$.

Since $\sum\frac{1}{a^2}=\frac{p^2-2q}{q^2}=\frac{16\left(p^2-2\cdot\frac{3p-1}{4}\right)}{(3p-1)^2}=\frac{8(2p-1)(p-1)}{(3p-1)^2}$,

$$\sum\frac{1}{a^3}=\frac{1}{q^3}\sum b^3c^3=\frac{p^3-3pq+3q^2}{q^3}=\frac{64\left(p^3-3p\cdot\frac{3p-1}{4}+3\cdot\left(\frac{3p-1}{4}\right)^2\right)}{(3p-1)^3}=\frac{4(16p^3-9p^2-6p+3)}{(3p-1)^3},$$

$$\frac{1}{q^4}\sum b^4c^4=\frac{p^4-4p^2q+4pq^2+2q^2}{q^4}=\frac{p^4-4p^2\cdot\frac{3p-1}{4}+2(2p+1)\cdot\left(\frac{3p-1}{4}\right)^2}{\left(\frac{3p-1}{4}\right)^4}=$$

$\frac{32(2p-1)(4p^3-p^2+2p-1)}{(3p-1)^4}$ then $F^2(a,b,c)=\frac{32(2p-1)(4p^3-p^2+2p-1)}{(3p-1)^4}-$

$$2\cdot\frac{4(16p^3-9p^2-6p+3)}{(3p-1)^3}+\frac{8(2p-1)(p-1)}{(3p-1)^2}=\frac{16(p^2-5p+2)^2}{(3p-1)^4}$$

and, therefore, $F(a,b,c) = \left|\dfrac{4(p^2 - 5p + 2)}{(3p-1)^2}\right|$.